Bose Condensation and Superfluidity in Finite Rotating Bose Systems

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There is a long standing problem about how close a connection exists between superfluidity and Bose condensation. Employing recent technology, for the case of confined finite Bose condensed systems in TOP traps, these questions concerning superfluidity and Bose condensation can be partially resolved if the velocity profile of the trapped atoms can be directly measured.

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1. Introduction

A long standing problem in the theory of superfluidity concerns the question of how close is the connection between Bose condensation and superfluidity[1]. For the case of superfluid ^4He, the Bose condensate fraction is quite small (if not virtually zero), yet the superfluid fraction goes to unity in the limit of low temperatures $T \to 0$. For the superfluid ^4He case, the observation of superfluidity is much more simple than the observation of Bose condensation. For the case of dilute Bose gases in atomic traps, it has been the case that the observation of Bose condensation appears more simple than the observation of superfluidity.

Let us pause to review the differences in theoretical definition between the Bose condensate fraction and superfluid fraction. For a Bose fluid in equilibrium, the reduced one particle density matrix is defined as

$$(\mathbf{r}|\rho|\mathbf{r}') = \big\langle \hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r})\big\rangle. \tag{1}$$

The one particle density matrix has a *maximum eigenvalue* N_{max} defined by

$$\int (\mathbf{r}|\rho|\mathbf{r}')\Psi(\mathbf{r}')d^3r' = N_{max}\Psi(\mathbf{r}), \tag{2}$$

where $\Psi(\mathbf{r})$ is the Bose condensate wave function. The condensate fraction may then be defined as

$$\eta_c = (N_{max}/N) \tag{3}$$

where N is the total number of atoms.

Now suppose that the fluid were in a rotating bucket[2]. Let $\mathbf{\Omega}$ denote the angular velocity of the bucket. If the fluid were to rotate as a rigid body, (as would any classical fluid), then the moment of inertia would be given by

$$I_{ij} = \int < \hat{\rho}(\mathbf{r}) > \big(r^2\delta_{ij} - r_ir_j\big)d^3r, \tag{4}$$

where $< \hat{\rho}(\mathbf{r}) >$ is the mean mass density of the fluid. If the fluid were to rotate as a superfluid, then the orbital angular momentum $< \hat{\mathbf{L}} >$ of the atoms would be somewhat smaller than that expected on the basis of rigid body rotation. Only the normal fluid rotates along with the bucket. This leads to the notion of a normal fluid fraction

$$\eta_n = \Big(\frac{\mathbf{\Omega}\cdot< \hat{\mathbf{L}} >}{\mathbf{\Omega}\cdot\mathbf{I}\cdot\mathbf{\Omega}}\Big). \tag{5}$$

The superfluid fraction in the Landau two fluid picture is defined as

$$\eta_s = 1 - \eta_n \tag{6}$$

Our purpose is to point out that *both* the condensate fraction η_c *and* the superfluid fraction η_s play a central role for Bose systems in TOP traps. The TOP trap homogeneous rotating contribution to the magnetic field serves to induce a laboratory rotating bucket. Thus, the question of whether or not a Bose condensate also produces superfluidity reduces to the question of whether or not the trapped fluid flow follows the rotation of the bucket.

If the fluid does not circulate in the TOP trap, then the fluid exhibits superfluid behavior. If the

fluid does rotate with a rigid body flow, closely following the bucket, then the fluid exhibits normal fluid behavior. If there is partial rotation of the fluid, then there is a finite superfluid fraction.

2. TOP Trap Magnetic Fields

The Hamiltonian for atoms in a TOP trap is thought to have the time varying form

$$H_{TOP}(t) = \sum_{1\le j\le N} h_j^{TOP}(t) + \sum_{1\le j<k\le N} v_{jk}, \qquad (7)$$

where v_{jk} is a two body potential,

$$h_j^{TOP}(t) = -\Big(\frac{\hbar^2}{2M}\Big)\nabla_j^2 - \gamma_j \mathbf{S}_j{\cdot}\mathbf{B}(\mathbf{r}_j, t) \qquad (8)$$

is a one body Hamiltonian, γ_j is the gyromagnetic ratio of the j^{th} atom, and $\mathbf{S}_j$ is the total spin of the j^{th} atom.

The time varying magnetic field in a TOP trap obeys

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_h(t) + \mathbf{B}_Q(\mathbf{r}), \qquad (9)$$

where the quadrapole field contribution $\mathbf{B}_Q(\mathbf{r})$ is given in terms of a static field gradient amplitude G,

$$\mathbf{B}_Q(\mathbf{r}) = G\big(\mathbf{r} - 3(\mathbf{n}\cdot\mathbf{r})\mathbf{n}\big). \qquad (10)$$

The homogeneous and time varying field $\mathbf{B}_h(t)$ rotates about (and is normal to) the unit vector $\mathbf{n}$; i.e. the "bucket angular velocity" is of the form $\mathbf{\Omega} = \Omega\mathbf{n}$. Furthermore, $\mathbf{n}\cdot\mathbf{B}_0 = 0$ and

$$\mathbf{B}_h(t) = \mathbf{B}_0\cos(\Omega t) + \mathbf{n}\times\mathbf{B}_0\sin(\Omega t). \qquad (11)$$

To understand why TOP trap magnetic field induces the Hamiltonian of a "rotating bucket", it is sufficient to introduce the total angular momentum, (i.e. total orbital plus total spin) of all N atoms

$$\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}} = \sum_j \big(\mathbf{r}_j{\times}\mathbf{p}_j\big) + \sum_j \mathbf{S}_j, \qquad (12)$$

where $\mathbf{p}_j = -i\hbar\nabla_j$. We employ the unitary rotation operator

$$U(t) = \exp\Big(-\frac{i\mathbf{\Omega}{\cdot}\hat{\mathbf{J}}t}{\hbar}\Big) = \exp\Big(-\frac{i\Omega\mathbf{n}{\cdot}\hat{\mathbf{J}}t}{\hbar}\Big), \qquad (13)$$

within the canonical transformation $H_{TOP}(t)\to\mathcal{H}$, as given by

$$\mathcal{H} = U^\dagger(t)H_{TOP}(t)U(t) - i\hbar U^\dagger(t)\Big(\frac{\partial U(t)}{\partial t}\Big). \qquad (14)$$

This transformation leads to a time independent Hamiltonian

$$\mathcal{H} = \hat{H} - \mathbf{\Omega}{\cdot}\hat{\mathbf{J}}. \qquad (15)$$

In the above Eq.(15), the time independent contribution $\hat{H}$ is given by

$$\hat{H} = \sum_{1\le j\le N} h_j + \sum_{1\le j<k\le N} v_{jk}, \qquad (16)$$

where

$$h_j = -\Big(\frac{\hbar^2}{2M}\Big)\nabla_j^2 - \gamma_j\mathbf{S}_j{\cdot}\tilde{\mathbf{B}}(\mathbf{r}_j). \qquad (17)$$

In the rotating frame, the magnetic field operator may be viewed as being static; i.e.

$$\tilde{\mathbf{B}}(\mathbf{r}) = \mathbf{B}_0 + G\big(\mathbf{r} - 3(\mathbf{n}\cdot\mathbf{r})\mathbf{n}\big). \qquad (18)$$

Thus Eqs.(15), (16), and (17) determine the typical "rotating bucket" form for the TOP trap Hamiltonian.

3. The Rotating Bucket Hamiltonian

As previously stated, The Hamiltonian in Eq.(15) is precisely of that type which is found for Bose fluids in rotating buckets. However, TOP traps have not always been viewed in this fashion. In the original experimental work on Bose condensates in TOP traps[3,4], one employed a time averaged adiabatic Hamiltonian in the analysis. Such an analysis may hold true if and only if the bucket rotates and the contained fluid does not rotate; i.e. such a model may hold true but only if the fluid acted as a perfect superfluid. The superfluid fraction in Eqs.(5) and (6) would have to obey $\eta_s = 1$ in order for the time averaged adiabatic Hamiltonian view to have any merit. In this time averaged view, the bucket rotates and the fluid does not rotate, a very perfect superfluid indeed.

It is very unlikely that such a perfect superfluid condition works for realistic experiments with the substantially high rotational velocities of TOP

traps. There is, in fact, no compelling physical motivation to ever time average the Hamiltonian. Time averaging is not the proper way to treat fluids in rotating buckets.

In more recent work[5–8], the time averaging was still being carried out but the rotational velocity was properly included only in so far as the spin degrees of freedom are concerned. The effective Hamiltonian of this most recent work reads[9] (in our notation) as

$$\mathcal{H}_{eff} = \hat{H} - \mathbf{\Omega}\cdot\hat{\mathbf{S}}. \qquad \text{(Wrong)}$$

The above equations is labeled as being wrong only by reason of employing a partial transformation (rotating only in spin angular momentum and not in orbital angular momentum). Such a transformation leaves an effective Hamiltonian which still depends on time by virtue of the quadrapole contribution to the magnetic field. Only after time averaging can the explicit time dependence be removed from $\mathcal{H}_{eff}$. The weakness of such an approach has been noted.

The proper method employs Eq.(15), in which the total angular momentum (spin plus orbital) generates the rotations. The resulting rotating frame Hamiltonian $\mathcal{H}$ is truly time independent, and properly includes the coupling to the orbital angular momentum of the atoms. This orbital angular momentum has been previously discussed [10] for TOP traps. The above Eq.(Wrong) should be replaced by

$$\mathcal{H} = \hat{H} - \mathbf{\Omega}\cdot\big(\hat{\mathbf{S}} + \hat{\mathbf{L}}\big). \qquad \text{(Correct)}$$

The above coupling is crucial for a proper analysis of rotational superfluidity. This is quite important for determining the nature of the connection between the superfluid fraction of Eqs.(5) and (6) and the Bose condensate fraction of Eq.(3).

Finally, in the adiabatic approximation, Eqs.(15), (16), (17) and (18) imply Hamiltonians of the form

$$\mathcal{H}_{adiabatic} = \hat{H}_{ad} - \mathbf{\Omega}\cdot\hat{\mathbf{L}} \qquad (19)$$

where

$$\hat{H}_{ad} = \sum_{1\le j\le N} h_j^{ad} + \sum_{1\le j<k\le N} v_{jk}, \qquad (20)$$

and where a simple adiabatic potential model yields

$$h_j^{ad} = -\Big(\frac{\hbar^2}{2M}\Big)\nabla_j^2 + V_j^{ad}(\mathbf{r}_j). \qquad (21)$$

The adiabatic potential is given by

$$V_j^{ad}(\mathbf{r}_j) = \big|M_{S,j}\gamma_j\big|\Big|\tilde{\mathbf{B}}(\mathbf{r}_j) + \frac{\mathbf{\Omega}}{\gamma_j}\Big|, \qquad (22)$$

where $\tilde{\mathbf{B}}(\mathbf{r})$ is defined in Eq.(18) and $M_{S,j}$ is an appropriate projection of the j^{th} atom total spin. Had we performed the adiabatic approximation before the rotational unitary transformation (as in previous work[10]), the adiabatic time independent rotational Hamiltonian would *still* have the form of Eq.(19). However, the adiabatic potential in Eq.(22) would be missing the spin induced $\mathbf{\Omega}/\gamma_j$ term. The subtle point is that the adiabatic limit and the rotational unitary transformation are not quite commuting processes. This leads to corrections in the adiabatic results which start at order $|\Omega/(\gamma_j B_0)|$.

4. Magnitudes

Let us consider the two rotational terms in

$$\mathcal{H} = \hat{H} - \mathbf{\Omega}\cdot\hat{\mathbf{J}} = \hat{H} - \mathbf{\Omega}\cdot\big(\hat{\mathbf{S}} + \hat{\mathbf{L}}\big). \qquad (23)$$

The spin term is at most $|\mathbf{\Omega}\cdot <\hat{\mathbf{S}}>| \sim N\hbar\Omega$. Furthermore, if there were but one single vortex line in the Bose condensed system, then the orbital term would also obey

$$|\mathbf{\Omega}\cdot <\hat{\mathbf{L}}>| \sim N\hbar\Omega. \qquad \text{(One Vortex Line)}$$

With one (or less) vortex line, the trapped Boson system would constitute an almost perfect superfluid in the sense of Eqs.(5) and (6).

On the other hand, the angular velocity of the TOP trap rotating bucket is high on the scale of simple estimates of the critical velocity, i.e.

$$\Omega >> \Omega_c \sim \Big(\frac{\hbar}{Mb^2}\Big) \qquad (24)$$

where b is a sensible length scale describing the size of the bucket. Employing such a simple estimate, one would expect many vortices. The formation of many vortices implies

$$|\mathbf{\Omega}\cdot <\hat{\mathbf{L}}>| >> |\mathbf{\Omega}\cdot <\hat{\mathbf{S}}>|. \qquad \text{(Many Vortices)}$$

Under no circumstances (other than vortex free perfect superfluidity) may the orbital angular momentum rotational coupling be regarded as a small perturbation on the spin rotational angular momentum coupling. This situation has been incorrectly described elsewhere[6].

Only for the case of a unit superfluid fraction $\eta_s \approx 1$ will there be zero vorticity at large bucket rotational velocities. In previous work[11], Rokhsar stated that from the work of Putterman, Kac, and Uhlenbeck[12], it follows that vortices in TOP traps will have cores which are not pinned. The vortices can thereby slip out of the fluid. Such depinning would leave behind a non-rotating condensate.

A non-rotating condensate together with a quickly rotating bucket means *perfect superfluidity*. For the continually rotating bucket situation with $\Omega >> \Omega_c$ there may be *stable vortices*. In judging the vortex stability one should use a statistical ensemble in which $\mathbf{\Omega}$ is fixed and angular momentum fluctuates. This is especially true when angular momentum is not conserved by virtue of the bucket potential which is not rotationally invariant[12]. It is unreliable for estimates of stability to employ the fixed angular momentum ensemble as in the work of Rokhsar[13]. The idea of an absolutely rotationally invariant bucket is a mathematical fiction not physically present in laboratories. It has long been known since the time of Newton that if you rotate a bucket fast enough, then the fluid will also rotate.

The angular velocity of the bucket can induce the vortices into a stable (rather than a metastable) configuration. The higher the TOP trap angular velocity, the more stable are the vortices. In this manner, rotating buckets containing a superfluid may yield a simulation (i.e. imitation) of a normal fluid. This requires stable vortex formation. In a fully Bose condensed system with $\eta_c \approx 1$, and with a mass current determined by the condensate wave function $\Psi(\mathbf{r})$, the formation of many vortices are the only means available for simulating the rigid body rotation within the fluid. Such an effects are well known for superfluid ^4He case even though $\eta_c(^4He) << 1$.

5. Conclusions

A fluid which undergoes rigid body rotation has a velocity field of the form

$$\mathbf{v}(\mathbf{r}) = \mathbf{\Omega} \times \mathbf{r}. \tag{25}$$

A classical fluid in a rotating bucket always[2] exhibits a velocity field as in Eq.(25). With the phase ϕ of the superfluid determined by the condensate wave function $\Psi = |\Psi| e^{iM\phi/\hbar}$, the superfluid velocity

$$\mathbf{v}_s(\mathbf{r}) = \nabla \phi(\mathbf{r}) \tag{26}$$

is irrotational and cannot duplicate the rotational velocity field of Eq.(25). The best that can be done is to create many vortices at a density which (on a coarse grained average) can simulate rigid body rotation and thus appear to be “normal”.

The final analysis is experimental and not theoretical. If, upon measuring the velocity profile of a TOP trap Bose system (via the Doppler shift) one observes a velocity field of the form in Eq.(25), then the fluid is “normal” for the high $\mathbf{\Omega}$, albeit in a possible Bose condensed state. If zero velocity field is detected in a rotating TOP trap, then the condensate is superfluid, and is in fact quite remarkable. It is unusual to observe a significant superfluid fraction in the regime $\Omega >> \Omega_c$ of Eq.(24).

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